density is usually much higher than the electron density computed at the mean local temperature. The importance of this in interpreting results of plasma experiments is all too obvious.

The question of whether the turbulent flow is in an equilibrium or frozen state remains important. This can be settled only if a comparison can be made of the relevant chemical relaxation times with the residence times of a fluid particle in a given "turb." Such a comparison cannot be made accurate within the present understanding of turbulent mixing. However, the observed importance of the entropy mode of fluctuations implies residence times large enough to be comparable with the flow times, such as the transit time between the body and the relevant station in the wake, for example; the probability of local equilibration is thus enhanced.

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Least-Squares Approach to Unsteady Kernel Function Aerodynamics

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A LEAST-SQUARES solution to the integral equation relating pressure and downwash in three-dimensional unsteady flow has distinct advantages over collocation solutions, particularly with regard to alleviating error between collocation points. Although the mathematical aspects of least-squares approximation to complex valued functions in higher dimensions are not well known,[1] the results are simple and straightforwardly applied. In view of this simplicity of application, it is hoped that the method employed herein will be useful to a wide variety of other engineering problems.

Basic Equations and Motivation

The problem of determining unsteady aerodynamic forces on finite span planforms leads to a singular homogeneous Fredholm integral equation between pressure and downwash.[2] Letting C_p be the perturbation pressure coefficient and w the downwash nondimensionalized by freestream velocity, the integral equation is

$$w(x, y) = \frac{1}{8\pi} \iint_s C_p(\xi, \eta)K(x - \xi, y - \eta;\ M, k)d\xi d\eta \quad (1)$$

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where S is the planform area, M and k are the Mach number and nondimensional reduced frequency, x, ξ and y, η are nondimensional chordwise and spanwise variables, and K is the complex valued kernel function. Workable expressions for K in subsonic[2] and supersonic[3] flow were given by Watkins et al. of NASA. Successful digital computer programs for predicting three-dimensional unsteady airloads based on an assumed mode approach were originally developed notably by Watkins, et al.[4,5] of NASA and Hsu, et al.[6] of Massachusetts Institute of Technology (MIT). Assuming that the pressure may be represented as a superposition of preselected pressure modes,

$$C_p(x, y) = \sum_i p_i(x, y)b_i \quad (2)$$

the coefficients b_i must satisfy the equation

$$w(x, y) = \sum_i [\iint_s p_i(\xi, \eta)K(x - \xi, y - \eta;\ M, k)d\xi d\eta]b_i \quad (3)$$

Equation (3) cannot, in general, be satisfied at all points on the planform. The forementioned computer programs require that Eq. (3) be satisfied at as many points as there are assumed modes; i.e., "downwash collocation." The resulting set of algebraic equations constitute a complex matrix equation

$$\{w(x_i, y_i)\} = [D_{ij}]\{b_i\} \quad (4)$$

where the matrix definitions are clear from Eq. (3).

An attempt was made to reduce error in the MIT program. The flow is regarded as two-dimensional, and collocation stations are chosen so that the error is zero *on the average*.[6] However, the flow is three-dimensional, particularly with modern low-aspect ratio surfaces. In addition, because "zero error on the average" means that equally bad errors of opposite sign are equivalent to two points of zero error, it is desirable to minimize the mean-square error over the entire surface.

Equation (4) guarantees only that the downwash is correct at collocation points; no guarantee is made between these points. Error between collocation points can seriously affect the solution, although no serious difficulties have occurred for all-movable flexible surfaces. However, there have been instances in supersonic flow where the airloads, due to the first and second vibration modes of surfaces cantilevered at the root, could not be calculated because of excessive error between collocation points. In addition, regardless of the surface and flow conditions, the solutions depend to some extent on collocation point locations, thereby creating a recurring question with regard to proper location of the points. Furthermore, it is sometimes desirable to leave out certain higher-order assumed pressure modes. However, the collocation solution necessitates an equal decrease in the number of collocation points, thereby increasing error. These difficulties would all be alleviated by a least-squares solution to Eq. (3). Moreover, it can be shown[1] that, in the least-squares case, ill conditioning is a direct measure of linear dependence of approximating functions, thereby providing, for the first time, a *usable* criterion for the adequacy of assumed pressure modes. This is particularly useful in supersonic flow where ill conditioning has been a problem.

The Least-Squares Approach

For notational simplicity, denote the integrals within brackets in Eq. (3) by $w_i(x, y)$. Then the problem becomes that of approximating the known downwash $w(x, y)$ by the downwash functions induced by the assumed pressure modes:

$$w(x, y) = \sum_1^n w_i(x, y)b_i \quad (5)$$

where n is the number of assumed modes. It is shown in Ref. 1 that the least-square error over the surface is obtained if and only if the following equation is satisfied by the coefficients:

$$\underset{n\times n}{[\textstyle\int_s \bar{w}_i(x, y)w_j(x, y)dxdy]}\ \underset{n\times 1}{\{b_i\}} = \underset{n\times 1}{\{\textstyle\int_s \bar{w}_i (x, y)w(x, y)dxdy\}} \quad (6)$$

provided that the coefficient matrix is nonsingular. The bars denote complex conjugation. It is also shown[1] that the coefficient matrix is always positive semidefinite and is positive definite and nonsingular if and only if the downwash functions due to assumed pressure modes are linearly independent over the surface.

Equation (6) is not yet in a form suitable for digital calculations. Because the integrals are quite complicated, numerical procedures are necessary. Using numerical integration for the coefficient matrix in Eq. (6), it follows that

$$\underset{n\times m}{[\bar{w}_i(x_j, y_j)]}\underset{m\times m}{\lceil W_i \rfloor}\underset{m\times n}{[w_j(x_i, y_i)]}\underset{n\times 1}{\{b_i\}} = \underset{n\times m}{[\bar{w}_i(x_j, y_j)]}\underset{m\times m}{\lceil W_i \rfloor}\underset{m\times 1}{\{w(x_i, y_i)\}} \quad (7)$$

where $(x_1, y_1), \ldots, (x_m, y_m)$ are m integration points and $\lceil W_i \rfloor$ is the diagonal matrix of integration weighting factors. One can see from Eq. (7) that the number of integration points must equal or exceed the number of assumed modes because the rank of the triple product cannot exceed the minimum of the ranks of its factors. Equation (7) can be stated in terms of more familiar matrices by observing that

$$D_{ij} = W_j(x_i, y_i) \quad (8)$$

Therefore, the final form of the least-squares equation is

$$\underset{n\times m}{[\bar{D}_{ij}]^T}\underset{m\times m}{\lceil W_i \rfloor}\underset{m\times n}{[D_{ij}]}\underset{n\times 1}{\{b_i\}} = \underset{n\times m}{[\bar{D}_{ij}]^T}\underset{m\times m}{\lceil W_i \rfloor}\underset{m\times 1}{\{W(x_i, y_i)\}} \qquad m \geqslant n \quad (9)$$

If $n = m$ and $[D_{ij}]$ is nonsingular, Eq. (9) for least-square error reduces to Eq. (4) for the collocation solution. The largest portion of computing time in existing programs is devoted to computing $[D_{ij}]$. Therefore, additional time required to perform the indicated matrix multiplications would be negligible. This leads to an important conclusion: existing collocation-based programs can be readily converted to solutions possessing minimum square error by increasing the number of points and replacing the coefficient matrix and right-hand side of Eq. (4) by their triple-product counterparts in Eq. (9). This includes the familiar procedures for real equations as a special case. It is seen that the conversion procedure is identical for mathematically similar problems involving complex valued equations.

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A Passive System for Determining the Attitude of a Satellite

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I. Introduction

A GROUP at the Applied Physics Laboratory has, during the past several years, developed "solar attitude" detectors.[1] In principle, these detectors are single solar cells placed on orthogonal axes of the satellite. From processing the telemetered voltage output from these cells, we can obtain the direction cosines of the satellite-sun vector.

Relative to the satellite-fixed axes defined by the solar detectors, we can place another set of sensors to measure the direction cosines of another vector in body-fixed coordinates. We have used three orthogonally placed fluxgate magnetometers to measure the direction cosines of the earth's magnetic field. With the direction cosines of two vectors and independent knowledge of the satellite position, then, we derive the attitude matrix of the satellite.

II. Analysis

We will define (in any convenient way) a set of orthogonal axes fixed in the satellite. Relative to this axis, we will then measure, at any particular instant, the required direction cosines.

Associated with the body-fixed axis is a "reference" coordinate system (e.g., inertial coordinates) such that if the relationship of the body axis to the reference system is available, we can then say that the attitude of the satellite is known. To make this statement explicit, we are given the components (r_1, r_2, r_3) of a vector of unit magnitude in body-fixed coordinates. This vector has a representation in another orthogonal coordinate system:

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \{A\} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \quad (1)$$

wherein $\{A\}$ is a (orthogonal) rotational matrix that transforms from body-fixed to, say, inertial coordinates. If $\{A\}$ can be obtained explicitly, then the attitude is known. In the system implemented by us, **r** is obtained from the solar attitude detectors, whereas **R** is computed from knowledge of the satellite orbit and the position of the sun.

It is intuitively clear that Eq. (1) is sufficient to determine the orientation of the satellite axes relative to the satellite sunline, excepting the degree of freedom involving rotation about this line. To determine all three degrees of freedom, it is sufficient to treat similarly any other linearly independent vector that can be resolved in satellite coordinates. For any vector **b**, linearly independent from **r**, we write

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \{A\} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (2)$$

In the system implemented by us, we compute **B** from the known position of the satellite and a mathematical model of the earth's magnetic field.[2] **b** is obtained from the fluxgate magnetometer telemetry results.

Equations (1) and (2) are then considered as a pair of simultaneous equations to be solved for the attitude matrix $\{A\}$. This solution is quite simple to achieve: Since $\{A\}$ transforms any vector in satellite coordinates, and since **r**

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